MATH 4571 - Lecture Notes

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Abstract

Studies the theory of vector spaces and linear maps and their applications, emphasizing deep understanding, proofs, and problem-solving.

Contents

1 LINEARITY

Definition 1.1

Linearity is the study of vector spaces (sets) and linear maps (transformations, linear functions).

Definition 1.2

A *linear system* is a system of equations that can be expressed in matrix form.

Example 1.3

The system

$$\begin{aligned} x_1 + x_3 &= 2 \\ 2x_1 + x_2 + 3x_3 - 2x_4 &= 5 \\ 3x_1 - 2x_2 + x_3 + 4x_4 &= 4 \end{aligned}$$

can be expressed as the following matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & -2 \\ 3 & -2 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$$

Definition 1.4

Euclidian space is a space of dimension $k \in \mathbb{N}$, expressed as

$$\mathbb{R}^k = \{(x_1, x_2, \ldots, x_k) : x_1, x_2, \ldots, x_k \in \mathbb{R}\}$$

$\mathbb{R}^k$ is also a field.

Definition 1.5

Vector spaces respect linear combinations. That is, if $x, y \in V$ for some vector space V, then $ax+by \in V$ where $a, b \in R$ are scalar coefficients.

Definition 1.6

An $m \times n$ matrix corresponds to a linear map/transformation

$$T : \mathbb{R}^n \to \mathbb{R}^m$$
$$T(x \in \mathbb{R}^n) = Ax \in \mathbb{R}^m$$

$\mathbb{R}^n$ is the domain and $\mathbb{R}^m$ is the codomain. The linear map is defined as

$$\begin{cases} T(x+y) = T(x) + T(y) \\ T(cx) = cT(x) \\ T(ax+by) = aT(x) + bT(y) \end{cases}$$

Not all maps are linear. Consider $T : \mathbb{R}^2 \to \mathbb{R}$ where $T(x) = ||x||$. This is not linear, since we can produce a counterexample that violates the properties of a linear map: $||(1,0)|| = 1 = ||(0,1)||$, but $||(1,1)|| = \sqrt{2} \neq ||(1,0)|| + ||(0,1)||$.

Definition 1.7

We define *reduced-row echelon form* (RREF) to be the matrix obtained from gaussian elimination, with additional constraints on *row echelon form*:

- The leading entry in each row is 1.
- Each column containing a leading 1 has zeroes in all its other entries.

Example 1.8

Given the following matrix under RREF

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

we can transform to equation

$$x = \begin{bmatrix} 2 - s \\ 1 - s + 2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

where the leading variables are x_1 and x_2, and the free variables $x_3 = s$ and $x_4 = t$.

2 SUBSPACES

January 17, 2024

Recall that a vector space $V, +, \cdot$ over some field F has closure under $+, \cdot$.

1. $+$ commutative
2. $+$ associative
3. $+$ identity 0
4. $+$ inverse $-v$
5. $(ab)v = a(bv)$
6. $(a+b)v = av + bv$
7. $c(v+w) = cv + cw$
8. $1 \in F : 1v = v$

F^S is the set of all functions $f : S \to F$. F^S is a vector space over F. We need to define addition and multiplication.

Let $f, g \in F^S$. We define addition to be $(f+g)(x) = f(x) + g(x)$ for $x \in S$.

We define multiplication to be $c \in F, f \in F^S$ to be $(cf)(x) = cf(x)$.

V1: How do we show that $(f+g)$ is the same as $(g+f)$? $(f+g)(x) = f(x)+g(x) = g(x)+f(x) = (g+f)(x)$

V2: Should also show associativity!

$$\begin{aligned} ((f+g)+h)(x) &= (f+g)(x) + h(x) \\ &= (f(x)+g(x)) + h(x) \\ &= f(x) + (g(x)+h(x)) \\ &= f(x) + (g+h)(x) \\ &= (f+(g+h))(x) \end{aligned}$$

V3: The zero function $0_f : S \to F, 0_f(x) = 0, \forall x \in S$.

V4: Additive inverse: $(-f)(x) = -(f(x))$. Note the inverse is $F \to S$.

The remaining vector space properties follow similarly.

Theorem 2.1

If W is a subset of a vector space V, then W is a subspace if and only if for any $v, w \in W$, $cv+w \in W$.

Proof. If W is a subspace, it is a vector space, and so closure of linear combinations implies that $cv + w \in W$.
If $cv + w \in W$, then W is a vector space. Because $W \subseteq V$, V1, V2, V5-V8 are automatically satisfied. So we need to show V3, V4.

- Closure of addition: Let $v, w \in W$. So $v + w = 1v + w \in W$.
- **V3**: $0 = (-1)v + v \in W$.
- Closure of multiplication: $x \in W, c \in F$ implies $cx = cx + 0 \in W$.
- **V4**: $-v = (-1)v + 0 \in W$.

□

Example 2.2: Examples of subspaces

- $V:\mathbb{R}^n$, $S:\{(x_1,\ldots,x_n):\mathbb{R}^n : x_1+\cdots+x_n=0\}$. We can see $cx+y=c(x_1,\ldots,x_n)+(y_1,\ldots,y_n)=cx_1+y_1,\ldots,cx_n+y_n$. And $c\underbrace{x_1+\cdots+x_n}_{0}+\underbrace{y_1,\cdots,y_n}_{0}=0$. So, S is a subspace.
- Consider $S=\left\{(t,t^2):t\in\mathbb{R}\right\}\subseteq\mathbb{R}^2$. Since $2\underbrace{(1,1^2)}_{\in S}=(2,2)\notin S$. So, S is *not* a subspace.

Remark 2.3: Subspace criterion.

Non-empty set W of vector space V is a subspace iff $cv+w\in W$.

In a vector space, there are two additive identities: the additive identity of the field and the additive identity of the vector space.

3 LINEAR SPAN

January 18, 2024

Definition 3.1: Span.

For the subset $S = \{v_1, \dots, v_k\} \subseteq V$, the $span(S)$ is the set of all linear combinations of $v_1, \dots, v_k$: $c_1 v_1 + \dots + c_k v_k$ for $c_i \in F$.

Remark 3.2

Observation: Each vector space has two trivial subspaces: the zero space and itself.

Definition 3.3

The *zero space* is simply $\{0\}$, a singleton set of the zero vector.

Definition 3.4

The span of the empty set is the zero space.

Theorem 3.5

The span of any subset S is a subspace. It is the smallest subspace among all subspaces containing S.

Proof. Let $S = \{v_1, \dots, v_k\} \subseteq V$. Let $v, w \in span(S)$. Then, v and w are linear combinations of $v_1, \dots, v_k$. So, $v = c_1 v_1 + \dots + c_k v_k$, and $w = d_1 v_1 + \dots + d_k v_k$. $cv + w$ is also a linear combination of $v_1, \dots, v_k$. So $cv + w \in span(S)$ and $span(S)$ is a subspace.
Let W be a subspace containing S. We want to show that $span(S) \subseteq W$. Let $x \in span(S)$. Then, x is a linear combination of $v_1, \dots, v_k$ and is in W since W is a subspace and contains all linear combinations of $v_1, \dots, v_k$. $\square$

Definition 3.6

If $V = span(S)$, then S is a *spanning set* (*generating set*) of V.

Remark 3.7

To determine if $v_1, \dots, v_k$ span V, ask if there exists some $v \in V$ such that there are no $c_1, \dots, c_k \in F^k$ where $v = c_1 v_1 + \dots + c_k v_k$.

4 ISOMORPHISMS

February 1, 2024

Theorem 4.1

$T : V \to W$ is 1 to 1 if and only if $kerT = \{0\}$.

Theorem 4.2

$T : V \to W$ is 1 to 1 and $v_1, \dots, v_n \in V$ is linearly independent if and only if $T(v_1), \dots, T(v_n) \in W$ is linearly independent.

Proof. If $c_1 T(v_1) + \dots + c_n T(v_n) = 0$, then $T(c_1 v_1 + \dots + c_n v_n) = 0$. Since T is 1 to 1, $kerT = \{0\}$, and $c_1 v_1 + \dots + c_n v_n = 0$. Since v_i's are linearly independent, c_i's are zero.
If $c_1 v_1 + \dots + c_n v_n = 0$, then $T(c_1 v_1 + \dots + c_n v_n) = T(0) = 0$. Since $T(v_1), \dots, T(v_n)$ are independent, c_i's are zero. □

Definition 4.3

$T \to W$ is an *isomorphism* if it is 1 to 1 and onto.

Definition 4.4

V and W are *isomorphic* if there exists an isomorphism.

Theorem 4.5

An isomorphism $T : V \to W$ has a unique inverse $T^{-1} : W \to V$ that is also an isomorphism.

Proof. Only need to show T^{-1} is linear. Let $c \in F$ and $x, y \in W$. Then, $x = T(v)$ and $y = T(u)$ for some $v, u \in V$. If T is linear, then $T(cv + u) = cT(v) + T(u) = cx + y$. And $T^{-1}(cx + y) = cv + u = cT^{-1}(x) + T^{-1}(y)$. □

Theorem 4.6

V, W are isomorphic if and only if $\dim(V) = \dim(W)$ for a finite dimension.

Proof. Since it's isomorphic, the kernel must be $\{0\}$ and so the nullity must be 0.
Conversely, suppose $\dim(V) = \dim(W)$. So, the basis of V is $v_1, \dots, v_n$ and the basis of W is $w_1, \dots, w_n$. Show that the following map is an isomorphism: $T : V \to W$ where $T(\underbrace{c_1 v_1 + \dots + c_n v_n}_{\in V}) = c_1 w_1 + \dots + c_n w_n$. Linear by theorem 2 of lesson 9.
To show 1 to 1, $T(c_1 v_1 + \dots + c_n v_n) = 0$ and $c_1 w_1 + \dots + c_n w_n = 0$, so the c_i's are zero, and $kerT = \{0\}$.
To show onto, let $w \in W$ so $w = c_1 w_1 + \dots + c_n w_n$. Clearly, $T(c_1 v_1 + \dots + c_n v_n)$ can produce w. □

5 MATRIX REPRESENTATIONS

February 5, 2024

Definition 5.1

Let $\beta : \{v_1, \ldots, v_n\}$ be an *ordered basis* of V. Then the *coordinate vector* of $v = c_1v_1 + \cdots + c_nv_n$ relative to β is $[x]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = (c_1, \ldots, c_n)$, where $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in F^n$.

Theorem 5.2

The map $\phi : V \to F^n$, where $\dim(V) = n$, defined by $\phi(v) = [v]_\beta$ is an isomorphism as ϕ is linear.

Proof. $\phi(cx + y) = [cx + y]_\beta = c[x]_\beta + [y]_\beta = c\phi(x) + \phi(y)$. □

Theorem 5.3

Let $T : V \to W$ be a linear map, and $\alpha = \{v_1, \ldots, v_n\}$ and $\beta = \{w_1, \ldots, w_n\}$ be ordered bases for V and W.

6 INNER PRODUCT SPACES

February 8, 2024

Definition 6.1

Let V be a vector space over F, where $F = \mathbb{R}$ or $F = \mathbb{C}$. An *inner product* on V is a function that assigns a scalar $< v, w >$ to each ordered pairs v, w such that for all vectors u, v, w and all scalars c,

- Linearity: $< cu + v, w >= c < u, w > + < v, w >$
- Conjugate symmetry: $< w, v >= < v,\bar{}w >$
- Positive-definiteness: $< v, v >\geq 0$; $< v, v >= 0$ only when $v = 0$.

7 ORTHOGONALITY

February 13, 2024

Definition 7.1: Cauchy-Schwarz Inequality

The *Cauchy-Schwarz Inequality* states $|\langle x, y\rangle| \leq \|x\|\|y\|$ where $-1 < \frac{\langle x,y\rangle}{\|x\|\|y\|} < 1$. So, the angle θ between two nonzero vectors x and y by $\cos\theta = \frac{\langle x,y\rangle}{\|x\|\|y\|}$ for $\theta \in [0, \pi)$.

Definition 7.2

Two vectors x and y are *orthogonal* when $\langle x, y\rangle = 0$ or the angle between them is $\frac{\pi}{2}$. A set of vectors S is an *orthogonal set* if every pair of vectors in S are orthogonal, and S is an *orthonormal* if in addition, all vectors in S has norm 1.

Theorem 7.3

An orthogonal set of nonzero vectors is linearly independent.

Proof. Let $v_1, \ldots, v_n$ be orthogonal vectors. If $c_1 v_1 + \cdots + c_n v_n = 0$, then for any k, $0 = \langle 0, v_k\rangle = \langle c_1 v_1 + \cdots + c_n v_n, v_k\rangle = c_1 \langle v_1, v_k\rangle + \cdots + c_n \langle v_n, v_k\rangle = c_k \langle v_k, v_k\rangle$. Since $\langle v_k, v_k\rangle \neq 0$, we have $c_k = 0$ for all k. ☐

Corollary 7.4

Any orthogonal set of n nonzero vectors in an n-dimensional space V is a basis of V (*orthogonal basis*).

Theorem 7.5

If $S = \{v_1, \ldots, v_n\}$ is an orthogonal basis of vector space V, then for any $x \in V$, $x = c_1 v_1 + \cdots + c_n v_n$, and $c_k = \frac{\langle x, v_k\rangle}{\|v_k\|^2}$.

Definition 7.6

If $w_1, \ldots, w_k$ is an orthogonal basis of a subspace W of an inner product space V, the *orthogonal projection* of $v \in V$ into W is

$$proj_w(v) = \frac{\langle v_1, w_1\rangle}{\|w_1\|^2} w_1 + \cdots + \frac{\langle v_k, w_k\rangle}{\|w_k\|^2} w_k$$

Definition 7.7

The *orthogonal complement* of W is the set $W^\perp$ of all vectors V which are perpendicular to every vector in W.

8 ORTHOGONAL BASIS

February 14, 2024

Theorem 8.1

Let $v_1, \dots, v_n$ be linearly independent vectors in an inner product space V, then for each $k = 1, \dots, n$, there is an orthogonal set $w_1, \dots, w_n$ in V which is a basis of $V_k = span(v_1, \dots, v_k)$.

Corollary 8.2

Every finite-dimensional inner product space V has an orthonormal basis; we simply normalize the vectors.

Corollary 8.3

Let $W = span(e_1, \dots, e_k)$ be a subspace of an inner product space V having an orthonormal basis $e_1, \dots, e_n$, then

- $e_{k+1}, \dots, e_n$ is an orthonormal basis of $W^{\perp}$.
- $V = W \oplus W^{\perp}$, and $\dim V = \dim W + \dim W^{\perp}$.

9 BEST APPROXIMATION

February 21, 2024

If we have some inconsistent linear system $Ax = b$, then b is not in the column space of A. The best we can do is find an *approximation* x^* such that Ax^* is as close as possible to b.

If W is a subspace of an inner product space V, for a vector $v \in V$, we are seeking a vector $w \in W$ such that $\|v - w\| \leq \|v - w'\|$ for every $w' \in W$.

Theorem 9.1

Let W be a finite-dimensional space of inner product space V and $v \in V$. If $w = \mathrm{proj}_W(v)$, then $\|v - w\| \leq \|v - w'\|$ for every $w' \in W$ with equality if and only if $w = w'$.

10 LINEAR FUNCTIONALS AND ADJOINTS

February 22, 2024

Definition 10.1

A linear map $T : V \to F$, $T(x) = \langle x, v \rangle$ that produces a scalar is *linear functional.* To show it is linear, $T(cx + y, v) = c \langle x, v \rangle + \langle y, v \rangle = cT(x) + T(y)$.

Theorem 10.2

Let V be a finite-dimensional inner product space and T be a linear functional on V, then there is a unique $v \in V$ such that $T(x) = \langle x, v \rangle$ for all $x \in V$.

Theorem 10.3: Adjoint

For any linear map $T : V \to W$, where V and W have finite-dimensional inner product spaces, there is a unique linear map $T^* : W \to V$ such that $\langle T(v), w \rangle_W = \langle v, T^*(w) \rangle_V$ for all $v \in V$ and $w \in W$; T^* is called the *adjoint* of T.

Theorem 10.4

If $\alpha = \{v_1, \ldots, v_n\}$ and $\beta = \{w_1, \ldots, w_m\}$ are orthonormal bases of finite-dimensional inner product spaces V and W respectively, and $T : V \to W$ is a linear map, then $[T^*]_\beta^\alpha = \left([T]_\alpha^\beta\right)^*$.

Theorem 10.5

If V and W are finite-dimensional inner product spaces, and $S : V \to W$ and $T : V \to W$ are linear maps, then

1. $(S + T)^* = S^* + T^*$
2. $(cT)^* = \overline{c}T^*$
3. $(ST)^* = T^* S^*$
4. $(T^*)^* = T$

Theorem 10.6

Suppose V is a finite-dimensional inner product space and $T^* : W \to V$ is the adjoint of $T : V \to W$, then $\ker T$ and $\operatorname{im} T^*$ are orthogonal complements in V.

11 EIGENVALUES AND EIGENSPACES

February 26, 2024

Definition 11.1

An *eigenvector* v of a linear map $T : V \to V$ is a nonzero vector such that $T(v) = \lambda v$ for some scalar called the *eigenvalue* associated with the eigenvector v.

Theorem 11.2

For any fixed eigenvalue λ of a linear map $T : V \to V$, the set E_λ of all vectors $v \in V$ satisfying $T(v) = \lambda v$ is a subspace of V. This space is called the λ-*eigenspace.*

Proof. For any $u, v \in E_\lambda$, $T(cu + v) = cT(u) + T(v) = c(\lambda v) + \lambda v = \lambda(cu + v)$, so $cu + v$ is in E_λ. □

12 DIAGONALIZATION

February 28, 2024

Given a linear operator $T : V \to V$, we want to find a basis of V so that the matrix of T is the simplest, diagonal, if possible, for diagonal matrices are the simplest matrices.

Definition 12.1

A linear operator $T : V \to V$ on a finite-dimensional space V is *diagonalizable* if there is a basis β of V such that the matrix $[T]_\beta^\beta$ is diagonal. A square matrix A is *diagonalizable* if it is similar to a diagonal matrix.

Theorem 12.2

If A and B are similar, that is, $B = Q^{-1}AQ$, then A and B have the same characteristic polynomial and therefore the same eigenvalues (with the same algebraic multiplicities). Moreover, v is an eigenvector of B with eigenvalue λ if and only if Qv is an eigenvector of A with eigenvalue λ. It follows that the eigenspaces of A and B have the same dimensions.

Proof. Recall that similar matrices have the same determinant. So, $\det(\lambda I - B) = \det(Q^{-1}(\lambda I)Q - Q^{-1}AQ) = \det(Q^{-1}(\lambda I - A)Q) = \det(\lambda I - A)$. If $Bv = \lambda v$, then $A(Qv) = AQv = QBv = Q(\lambda v) = \lambda(Qv)$. If $A(Qv) = \lambda(Qv)$, then $Bv = Q^{-1}AQv = Q^{-1}\lambda(Qv) = \lambda v$. □